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FAST TRACK COMMUNICATION

The presence and lack of Fermi acceleration in nonintegrable billiards

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Abstract
The unlimited energy growth (Fermi acceleration) of a classical particle moving in a billiard with a parameter-dependent boundary oscillating in time is numerically studied. The shape of the boundary is controlled by a parameter and the billiard can change from a focusing one to a billiard with dispersing pieces of the boundary. The complete and simplified versions of the model are considered in the investigation of the conjecture that Fermi acceleration will appear in the time-dependent case when the dynamics is chaotic for the static boundary. Although this conjecture holds for the simplified version, we have not found evidence of Fermi acceleration for the complete model with a breathing boundary. When the breathing symmetry is broken, Fermi acceleration appears in the complete model.

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The growth of a particle's energy due to its interactions with a time-dependent potential was first noticed by Enrico Fermi [1] in the description of the acceleration mechanism of cosmic ray particles by magnetic fields [2]. This mechanism can be modeled by a classical light particle colliding elastically with time varying heavy boundaries (walls). The main point is to prove the existence of unlimited growth of the energy (Fermi acceleration) in this situation. Fermi ideas have been applied in different contexts such as atom optics [3], plasma physics [4], atomic physics [5] and astrophysics [6]. The one-dimensional problem of a particle moving between a fixed wall and an oscillating one (Fermi–Ulam model) was first examined by Ulam [7], but no acceleration was found. This result was later explained by the existence of invariant spanning curves in the phase space that prevents the unlimited growth of the energy [8–10]. It was soon noticed, as first studied by Hammersley [11] in a model in which the phase of the moving wall is randomly chosen at the collision time, that stochastic versions of the model could present an unlimited growth of the particle average energy. However, the most interesting question whether Fermi acceleration can result from the dynamics without any stochastic component, as firstly considered by Ulam [7], remains open and has been approached both from the numerical (simulation of the dynamics) and the analytical points of view. Sometimes, simplified versions of the models [9, 12–15] are introduced in order to speed up the simulations and develop analytical treatments for these nonlinear problems. In these simplified versions, the displacement of the moving wall is neglected but, in the collisions of the particle with it, the full time dependence of the momentum exchange between the particle and wall is taken into account. Recently, the stochastic Fermi–Ulam model was revisited [16] and it was shown that the simplified version leads to an underestimation of the particle's Fermi acceleration.

In two-dimensional billiards, a particle moves freely inside a given region of the plane and undergoes elastic collisions at the boundary. These models have been widely used in the investigation of classical and statistical mechanics [17], and in quantum physics [18]. Billiards with moving boundaries (time-dependent billiards) are a natural two-dimensional generalization of the original Fermi accelerator model. The acceleration in time-dependent billiards was investigated in different models including both integrable and nonintegrable or chaotic models. No acceleration was found for integrable billiards such as the breathing circle [19] or time-dependent ellipses [20]. In fact, as the breathing circle conserves angular momentum it can be reduced to the one-dimensional model and thus the existence of rotational invariant curves in the phase space prevents the unlimited growth of the energy. Such invariant curves also seem to exist in the time $\times$ energy phase space of the ellipse. On the other hand, Fermi acceleration was observed in a Lorentz-type dispersing billiard with both periodic and stochastic time-dependent boundaries, and in the stadium with periodically oscillating boundaries [21, 22]. Both models are known to have chaotic behavior in the static case. All those results enforce the belief that Fermi acceleration should be observed in periodically time-dependent perturbed chaotic trajectories, where the random behavior in position space plays the role of the random time-dependent perturbation of the one-dimensional model. This was formally conjectured by Loskutov *et al* [22] as 'A chaotic dynamics for the static billiard is a sufficient condition for the presence of Fermi acceleration in the billiard with time-dependent boundary.' Further works on the stadium with strong chaotic properties and on the near-rectangle billiard [23], and on the annular billiard [24, 25] were in agreement with the above conjecture, which is referred to as the LRA conjecture.

In this work we study a billiard with a parameter-dependent boundary. As this parameter is changed, the shape of the boundary is modified and pieces of the boundary can change from focusing to dispersing and *vice versa*. Introducing a time-dependent perturbation of the parameters, the boundary changes periodically in time. Although our numerical results for the simplified version are in agreement with the above conjecture, the complete model with breathing boundary does not have Fermi acceleration. When the breathing symmetry is broken, the system has unlimited growth of the particle average energy (velocity).

We consider the billiard inside a plane and closed curve Γ given in polar coordinates by $r(\theta) = 1 + \epsilon \cos 2\theta$, for $0 \leqslant \theta < 2\pi$. Here $\epsilon \in [0, 1)$ is a parameter controlling the shape of the curve, for instance Γ is a circle if $\epsilon = 0$. It is easy to verify that Γ is strictly convex, i.e. its curvature is positive, if $\epsilon < 0.2$ implying that we have a focusing billiard. If $\epsilon = 0.2$, Γ has isolated points of null curvature and if $\epsilon > 0.2$ it has nonconvex (dispersing) pieces of the boundary. Some of these curves are displayed in figure 1. For $\epsilon \geqslant 1$ the curve has self-intersections and is no more simple.

As θ parameterizes the curve Γ, the position of the particle at the nth impact is determined by θ_n. Defining the variable α_n as the angle of the velocity vector $\vec{v}_n$ with the tangent vector of

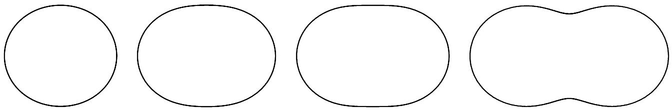

Figure 1. Curve Γ for $\epsilon = 0.05$, 0.15, 0.2 and 0.4.

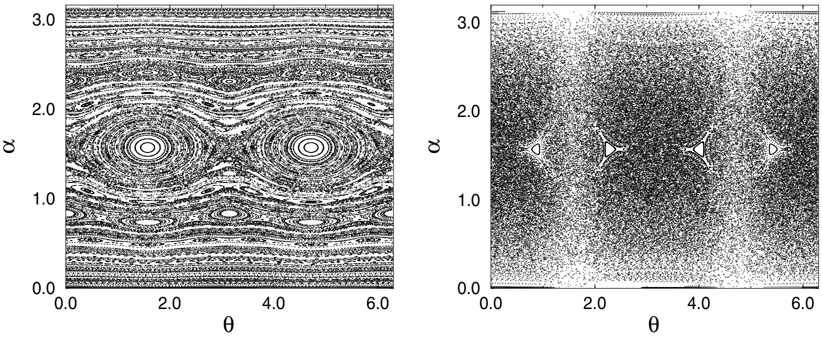

Figure 2. Static phase space for $\epsilon = 0.05$ (left) and $\epsilon = 0.4$ (right).

Γ at θ_n, the outgoing trajectory is then completely characterized by the pair (θ_n, α_n). It is clear that once this trajectory is given, the next impact point, and then θ_{n+1}, is defined by an implicit equation. Reflection law gives α_{n+1}. Note that the modulus of the velocity v_n is conserved. So the dynamics of the billiard inside Γ is described by a two-dimensional map between impacts: $(\theta_n, \alpha_n) \rightarrow (\theta_{n+1}, \alpha_{n+1})$. The corresponding phase space (figure 2) has a rich structure: there are KAM islands surrounded by a chaotic sea. For a strictly convex boundary ($\epsilon < 0.2$) there are also invariant spanning curves, which are destroyed when the boundary has concave pieces ($\epsilon > 0.2$). The billiard inside Γ is nonintegrable if $\epsilon > 0$.

A time-dependent version of this billiard is given by

$$r(t, \theta) = 1 + \eta_2 \cos(t) + \epsilon(1 + \eta_1 \cos(t)) \cos(2\theta), \quad (1)$$

where all quantities are dimensionless. For each t the equation above defines a curve $\Gamma(t)$. If $\eta_2 = \eta_1$ we have a breathing billiard because in this case $r(t, \theta) = [1 + \eta_1 \cos(t)](1 + \epsilon \cos(2\theta))$, with the term between square brackets representing a time-dependent global change of scale. If $\eta_2 \neq \eta_1$ the billiard changes its shape. The dynamics of a time-dependent billiard can be fully described by a four-dimensional map $(\theta_n, \alpha_n, t_n, v_n) \rightarrow (\theta_{n+1}, \alpha_{n+1}, t_{n+1}, v_{n+1})$ [26]. Besides the usual coordinates, one has to take into account time and energy variables, here represented by the instant t_n at which the nth collision occurs and v_n the modulus of the velocity after this collision. Given θ_n and t_n the initial position $\vec{r}_p(t_n) = (x, y) = r(t_n, \theta_n)(\cos\theta_n, \sin\theta_n)$ of the particle on $\Gamma(t_n)$ is determined. The unitary tangent vector of the curve, at the initial position, is denoted by $\hat{\tau}_n = (\cos(\phi_n), \sin(\phi_n))$ and the unitary inward normal vector by $\hat{\eta}_n$. Here the slope of the tangent (see figure 3) is given by $\tan(\phi_n) = [(\mathrm{d}y/\mathrm{d}\theta)/(\mathrm{d}x/\mathrm{d}\theta)]_n$. The velocity vector $\vec{v}_n = v_n(\cos(\phi_n + \alpha_n), \sin(\phi_n + \alpha_n))$, and so the straight movement $\vec{r}_p(t) = \vec{r}_p(t_n) + (t - t_n)\vec{v}_n$ are then specified by α_n and v_n. To find numerically the next impact, the position of the particle is evaluated for each small time increment until one can decide that the particle has reached the moving boundary again at time t_{n+1} and position defined by θ_{n+1}, i.e., $|\vec{r}_p(t_{n+1})| = r(t_{n+1}, \theta_{n+1})$ is satisfied. The boundary velocity at the $(n + 1)$th impact point is given by $\vec{u}_{n+1} = \dot{r}(t_{n+1}, \theta_{n+1})(\cos(\theta_{n+1}), \sin(\theta_{n+1}))$,

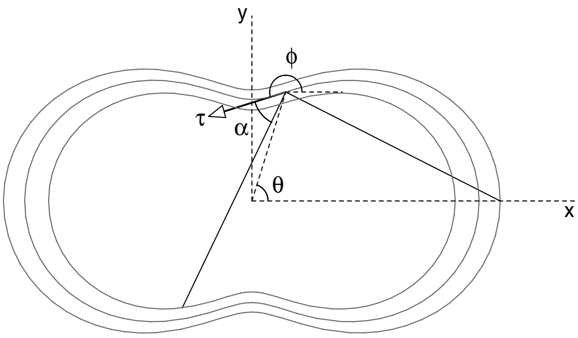

Figure 3. Trajectory of the particle (solid lines) and three consecutive collisions with the moving boundary. For the second impact, it is displayed the polar angle θ, the angle ϕ between the unitary tangent vector $\hat{\tau}$ and the x-axis, and the angle α between $\hat{\tau}$ and the particle velocity after the impact.

where $\dot{r}$ is the time derivate of r in equation (1). In the referential frame moving with this velocity, i.e., in which the point of impact is at rest and the particle moves with velocity $\vec{v}_n - \vec{u}_{n+1}$ the collision is elastic, implying that the normal component of the velocity of the particle is reversed and the tangential one is conserved. So in the original referential frame we have $\vec{v}_{n+1} \cdot \hat{\tau}_{n+1} = \vec{v}_n \cdot \hat{\tau}_{n+1}$ and $\vec{v}_{n+1} \cdot \hat{\eta}_{n+1} = (-\vec{v}_n + 2\vec{u}_{n+1}) \cdot \hat{\eta}_{n+1}$. This defines v_{n+1} and the new direction of motion α_{n+1}. We will use the denomination *geometrical phase space* to indicate the two-dimensional subspace (θ, α) of the four-dimensional phase space. Note that in the one-dimensional Fermi model, the phase space has dimension 2 and is given by the set (t, v).

To investigate the presence of Fermi acceleration in this model we analyze the behavior of the energy E as a function of the number of collisions n. We denote by

$$E(n) = \frac{1}{n+1} \sum_{j=0}^{n} E_j, \qquad \text{with} \qquad E_j = v_j^2/2,$$

the average along an orbit (Birkoff average). This average may depend on the initial condition. Averages over an ensemble of initial conditions are denoted by $\langle \cdots \rangle$.

Fixing $\epsilon = 0.4$ and choosing η_1 and η_2 small enough we guarantee that for each time, the billiard $\Gamma(t)$ has dispersing components and is strongly chaotic (see figure 2 (right)). In figure 4, it is shown the evolution of $E(n)$ of one initial condition for different values of η_1 and η_2. The initial condition corresponds to a chaotic orbit in the geometrical phase space. For the breathing case ($\eta_1 = \eta_2 = 0.1$) we can see that the energy seems to reach a constant value. On the other hand, when $\eta_1 \neq \eta_2$, the billiard changes its shape and the energy increases. Even a slightly deviation from the breathing case, for example $\eta_1 = 0.1$ and $\eta_2 = 0.11$, is sufficient for the energy to increase. This behavior is typical in the sense that it is independent of the initial conditions $(\theta_0, \alpha_0, t_0, v_0)$ as far as the trajectory stays in the chaotic region of the geometrical phase space. In particular, we have not observed any dependence on the initial velocity. More precisely, for the breathing model, no acceleration was detected at any initial velocity ranging from 1 to 100 and so we have not observed the existence of a critical velocity, above which the acceleration is present as in [23]. The generic behavior of these orbits is confirmed by averaging over 200 initial conditions along 10^7 collisions. Since the chaotic region spreads over almost all geometrical phase space, we can evaluate averages by choosing random values for θ_0 and α_0. In figure 5 we have plotted $\langle E(n) \rangle$ as a function of n. We can see for $\eta_1 \neq \eta_2$ that asymptotically the averages

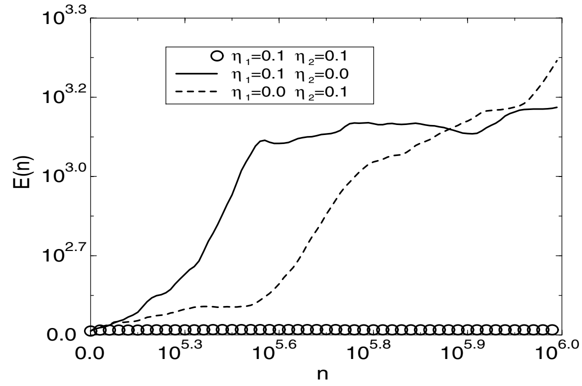

Figure 4. Plot of the average energy E along an orbit as a function of n for the complete time-dependent billiard with $\epsilon = 0.4$, 10^5 collisions and the initial condition ($\theta_0 = 0, \alpha = 0.4\pi$, $v_0 = 5$). Note that the energy seems to reach a constant value (around 32) for the breathing case.

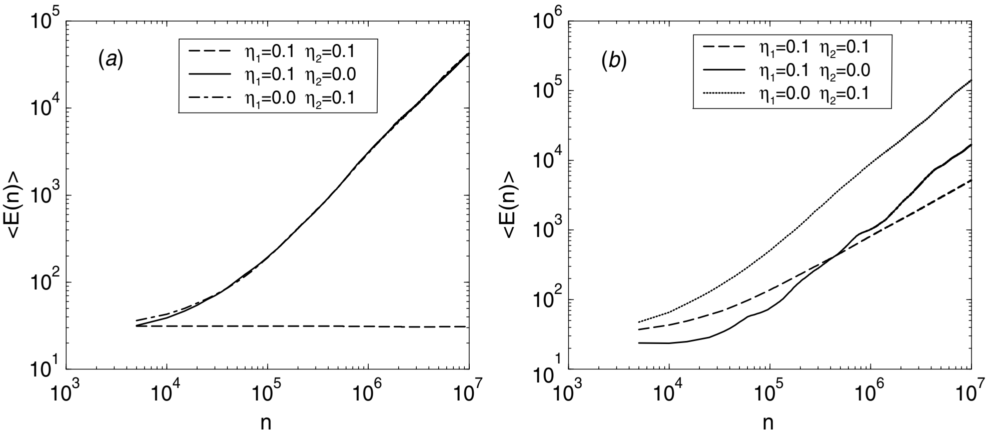

Figure 5. Log–log plots of $\langle E(n)\rangle$ as a function of n for the time-dependent billiard with $\epsilon = 0.4$, $v_0 = 5$, 10^7 collisions and 200 randomly values of θ_0 and α_0 in the averages. (a) The complete model; (b) the simplified model.

behave as $\langle E(n)\rangle \propto n^{\delta}$. Using the ordinary least-squares regression method we obtain for the case $\eta_1 = 0.1, \eta_2 = 0.0$ that $\delta = 1.165(5)$ with very good correlation coefficients. We found similar values for the other case ($\eta_1 = 0.0, \eta_2 = 0.1$), namely $\delta = 1.179(6)$. $\langle E_n\rangle$ exhibits similar behavior with a slightly different exponent. The average of the modulus of the velocity $\langle v(n)\rangle$, obviously, behaves as $\langle v(n)\rangle \propto n^{\beta}$ with $\beta = 0.587(4)$ and $\beta = 0.591(3)$ for the cases ($\eta_1 = 0.1, \eta_2 = 0.0$) and ($\eta_1 = 0.0, \eta_2 = 0.1$), respectively. We have also changed the initial velocity v_0 and/or the initial phase of the boundary with no significant effect on the results. The conclusion also holds if η_1 and η_2 have one of the values 0.0 and 0.3. Moreover, we have also introduced a difference of phase and/or frequency in the oscillations of the boundary; more specifically, equation (1) was replaced by the more general expression $r(t,\theta) = 1 + \eta_2 \cos(wt + \gamma_2) + \epsilon[1 + \eta_1 \cos(t + \gamma_1)] \cos(2\theta)$. Still, there is no significant influence of the new parameters in our conclusions. However, in the simplified version we have found that the energy increases even in the breathing case. The apparently unlimited growth of $\langle E(n)\rangle$ can be seen in figure 5(b). The values of the exponent when $\eta_1 \neq \eta_2$ is approximately equal to that of the complete model and for $\eta_1 = \eta_2 = 0.1$ (simplified breathing model) we have obtained $\delta = 0.800(2)$, indicating that even if the growth is less accentuated, it is still present.

When $\epsilon = 0.05$ the static billiard is everywhere focusing, i.e., its boundary is strictly convex. The phase space has regions with regular and chaotic dynamics, invariant spanning curves and large KAM islands (figure 2-left). Such features are characteristic of a near-integrable billiard. If we choose η_1 and η_2 to take one of the two values 0.0 or 0.1, the boundary curve $\Gamma(t)$ is strictly convex for all t, i.e. the billiard is everywhere focusing and large KAM islands associated with a period 2 orbit are always present in the geometric phase-space, although their size depends on time. Nevertheless, we can choose initial conditions which seem to stay in the (narrow) chaotic region around these islands. We have analyzed the behavior of the energy by taking averages upon such orbits and results similar to the strongly chaotic case described above were obtained. In the breathing case, we have not detected Fermi acceleration, while if the billiard changes shape, it was observed. In both cases, orbits which seem to remain trapped in elliptic islands and have a kind of regular dynamics do not show any acceleration.

This work is focused on the search of Fermi acceleration in deterministic two-dimensional nonintegrable billiard models, with both regular and chaotic regions. More precisely, we proposed a time-dependent billiard model, in which we can analyze the effect of the periodic movement of the boundary on *chaotic orbits* and test the LRA conjecture. Although we have made much numerical work, with many different values of the parameters involved, we choose to present here only two representative cases: a near integrable billiard and a strongly chaotic one. In one sense, our numerical results give a positive answer to the question of if it is possible to give unlimited energy, through collisions with a moving wall, to a particle undergoing chaotic motion. However, one has to observe that the exponent characterizing the energy average growth $\langle E(n)\rangle \propto n^{\delta}$ is larger than the one found for the stochastic model ($\delta = 1.0$); so the origin of the Fermi acceleration still remains to be understood.

On the other hand, we present evidence that chaotic motion is not a sufficient condition for the existence of Fermi acceleration, as the mean energy does not increase when the boundary oscillates in time in such a way that its shape is preserved (breathing case). Moreover, our work also puts in evidence the qualitative difference between the simplified model, in which we neglect the boundary displacement, and the complete model, as for the first, average energy growth in chaotic orbits is observed even in the breathing case. This may indicate that Fermi acceleration can be very sensitive to the way in which the boundary moves and consequently that the problem may be even more complex than expected.

At first sight, the result of the breathing case for the complete model contradicts the results in the literature. However, let us point out that some results [21–24] were in fact obtained for simplified versions, and the Fermi acceleration found for the complete annular breathing billiard [25] is due to the fact that the oscillating mode is not a real breathing one, because the billiard changes its shape during evolution. Unfortunately we have not found any consistent explanation for the lack of Fermi acceleration in the breathing case. Certainly new simulations in other breathing models and analytical arguments are welcome to explain this somehow unexpected result.

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